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REGULARITY OF POISSON QUADRATIC STOCHASTIC OPERATOR GENERATED BY 2-PARTITION OF A SINGLETON

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Abstract: The theory of quadratic stochastic operator (QSO) was developed by Bernstein in 1924 when he presented about population genetics. The research on QSO is still ongoing as researchers have not yet studied various classes, conditions, and measures. In this paper, we introduce a new class of Poisson QSO generated by the 2-partition of a singleton defined on the set of all integers state space. We illustrate the trajectory behaviour of the constructed QSO by cases. Lastly, we show that it is a regular transformation for some parameters' values.

Keywords: Quadratic stochastic operator, Poisson distribution, regular transformation.

Introduction

Bernstein first proposed the idea of the quadratic stochastic operator (QSO) in 1924 (Bernstein, 1924). This concept connects the mathematics field to the population of heredity. For more than 90 years, this idea has been developed in various fields, including but not limited to mathematics and biology. QSO is classified as a nonlinear operator and the key issue in nonlinear operator theory is to investigate its asymptotical behaviour.

A discrete dynamical system describes the time-dependent evolution of the system's state which this time evolution can be associated with a QSO. The QSO captures the probabilistic transitions between states, allowing for the analysis of the long-term behaviour of the system. QSO is important as it is used in various domains, including pure and social sciences. In epidemiology, for example, QSO can mimic disease propagation, assess intervention options, and predict the course of infectious diseases. While in the social sciences, QSO can help researchers examine social dynamics, opinion formation, and decision-making processes within populations. Furthermore, the study of dynamical systems appears to have a strong relationship to Earth system management because it is employed in Earth system science to anticipate safety, flexibility, and desirability (Heitzig *et al.,* 2016). Thus, the study of dynamics in QSO is important to sustain the knowledge and development for a better future.

Most researchers are fascinated with investigating the QSO on the lower dimension simplex. For instance, Saburov and Yusof (2018) examined the uniqueness of fixed points of QSO on 2D simplex, while Hardin and Rozikov (2018) studied a quasi-strictly non-Volterra QSO where they considered a four-parameter family of non-Volterra operators defined on 2D simplex and they proved that this operator has a unique fixed point but with one exception. Besides, Ikrom Kizi Mamadova (2022) continued the research on strictly non-Volterra QSO, in which she demonstrated and proved the uniqueness of the fixed point of this operator on 2D simplex. Mukhaedov *et al.* (2023) presented a class of $\xi^{(a)}$ –QSO on 2D simplex to investigate the algebraic properties of the genetic algebras associated with $\xi^{(a)}$ –QSO as well as their dynamics behaviour.

Meanwhile, a few studies focus on the QSO in infinite state space. The research on the continual state space of such operators can be seen from Gaussian QSO and Lebesgue QSO (Ganikhodjaev & Hamzah, 2015a, 2015b; Ganikhodjaev *et al.,* 2016; Hamzah *et al.,* 2022). They proved that Gaussian QSO and Lebesgue QSO are regular and nonregular depending on the values of the parameters. Additionally, there are some publications focus on infinite dimensional QSO, such as "On Lyapunov Functions for Infinite Dimensional Volterra Quadratic Stochastic Operators" and "Linear Lyapunov Functions on Infinite Dimensional Volterra Operators" (Mukhamedov & Embong, 2018; 2019). The authors presented constructions of the Lyapunov functions for such operators, aiming to estimate their limiting points. Furthermore, the studies titled "Infinite Dimensional Orthogonality Preserving Nonlinear Markov Operators" and "Projective Surjectivity of Quadratic Stochastic Operators on L^1 and its Application" represent additional discoveries that stemming from the study of the infinite-dimensional QSO (Mukhamedov *et al*., 2021; Mukhamedov & Embong, 2021).

On the other hand, the research on QSO on countable state space named Geometric QSO has been presented by Karim *et al.* (2019) where they have proved that a limit behaviour of Geometric QSO generated by 2-partition of a singleton is a regular transformation. The next paper demonstrated that a Geometric QSO generated by a 2-partition of infinite points also involves regular transformation (Karim *et al*., 2020). In addition, a Geometric QSO generated by 2-partition with $A_1 = |2|$ is a regular transformation, while a Geometric QSO with two and three different parameters can be either regular or nonregular depending on the values assigned to the parameters (Karim *et al.,* 2022a; 2022b).

Next, the dynamics and behaviour of another class of countable case of QSO, which is Poisson QSO can be seen in (Ganikhodjaev & Hamzah, 2014; Hamzah & Ganikhodjaev 2016b; Karim *et al*., 2021b). The authors proved the regularity of these operators. The research findings mentioned above motivate an exploration into a new class of Poisson QSO generated by 2-partition defined on the set of all integers. This inquiry is sparked by the lack of prior investigations into this specific QSO on the set of all integers. Consequently, this paper introduces a new class of Poisson QSO generated by 2-partition defined on the set of all integers of a singleton. Significantly, there will be various cases that can be considered to investigate their trajectory behaviour. In the following part, we will provide the idea of a quadratic stochastic operator on the set of all probability measures and a detailed definition of the Poisson QSO.

Methodology

Preliminaries

Let (X, F) be a measurable space, where X is a state space and F is σ –algebra on X and $S(X, F)$ is a set of probability measures on (X, F). Let $\{P(x, y, A): x, y \in X, A \in F\}$ be a family of functions on $X \times X \times F$ that satisfy the following conditions:

(i) $P(x, y,.) \in S(X, F)$ for any fixed $i, j \in X$,

(ii) $P(x, y, A)$ regarded as a function of two variables x and y with fixed $A \in F$, is a measurable function on $(X \times X, F \otimes F)$, and

(iii) $P(x, y, A) = P(y, x, A)$ for any $x, y \in A, A \in F$.

We consider a nonlinear transformation, which is called a quadratic stochastic operator (QSO) $V: S(X, F) \rightarrow S(X, F)$ defined by

$$(V\lambda)(A) = \int_X \int_X P(x, y, A) d\lambda(x) d\lambda(y), \quad (1)$$

where $A \in F$ is an arbitrary measurable set.

Poisson Quadratic Stochastic Operator

Let $\{P(i, j, k): i, j, k \in X\}$ be a family of functions defined on $X \times X \times F$, which satisfy these conditions:

(i) $P(i, j,.)$ is a probability measure on (X, F) for any fixed $i, j \in X$,

(ii) $P(i, j, k) = P(j, i, k) = P_{i,j,k}$, where $k \in X$ for any fixed $i, j \in X$.

In the discrete case, QSO in (1) on a measurable space (X, F) is defined as;

$$V\mu(k) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j), \quad (2)$$

where $k \in X$ for arbitrary measure $\mu \in S(X, F)$.

Recall that a Poisson distribution P_λ with positive parameter $\lambda > 0$ is defined on the set of non-negative integers $X = \{0, 1, ...\}$ as follows;

$$P_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!},$$

for any $k \in X$.

Definition 1 (Ganikhodjaev & Hamzah, 2014): A QSO V in (2) with $X = \{0, 1, ...\}$ is called a Poisson QSO if for any $i, j \in X$, the probability measure $P(i, j,.)$ is the Poisson distribution with $P_{\lambda(i,j)}$ positive parameter $\lambda(i, j)$ where $\lambda(i, j) = \lambda(j, i)$.

Definition 2 (Ganikhodzhaev *et al.,* 2011): A measure $\mu \in S(X, F)$ is called a fixed point of QSO V if $V\mu = \mu$.

Definition 3 (Ganikhodzhaev *et al.*, 2011): A QSO V is called regular if for any initial point $\mu \in S(X, F)$, there exists a limit such that;

$$\lim_{n\to\infty} V^n(\mu).$$

The dynamics and behaviour of Poisson QSO can be found in (Ganikhodjaev & Hamzah, 2014; Hamzah & Ganikhodjaev, 2016b; Karim *et al.,* 2021b). The authors have demonstrated that these operators are regular. In this paper, we intend to study a new class of Poisson QSO generated by 2-partition defined on a set of all integers and investigate their trajectory behaviour.

Poisson QSO Generated by 2-Partition

In this section, we discuss Poisson QSO generated by a 2-partition. Let $X = \mathbb{Z}$, where $\mathbb{Z}$ is a set of all integers. Recall that a partition of (X, F) is a disjoint collection of elements of F whose union is X. The finite partitions which are measurable k-partition will be denoted as $\xi = \{A_1,\ldots, A_k\}$.

We let $\xi = \{A_1, A_2\}$ be a measurable 2-partition of state space $\mathbb{Z}$, where $A_1 \subset \mathbb{Z}$, $A_2 \subset \mathbb{Z}\backslash A_1$, and $\zeta = \{B_1, B_2, B_3\}$ be a corresponding partition of space $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ where $B_1 = A_1 \times A_1$, $B_2 = A_2 \times A_2$ and $B_3 = (A_1 \times A_2) \cup (A_2 \times A_1)$.

Let a Poisson QSO such that a family of functions $\{P_{i,j,k}: i, j, k \in \mathbb{Z}\}$ defined as follows:

For $k = 0$,

$$P_{ij,0} = \begin{cases} e^{-\lambda_1} & \text{if } (i,j) \in B_1, \\ e^{-\lambda_2} & \text{if } (i,j) \in B_2, \text{ and} \\ e^{-\lambda_3} & \text{if } (i,j) \in B_3, \end{cases}$$

for $k > 0$,

$$P_{ij,k} = \begin{cases} \alpha e^{-\lambda_1}\dfrac{\lambda_1^{\,k}}{k!} & \text{if } (i,j) \in B_1, \\ \alpha e^{-\lambda_2}\dfrac{\lambda_2^{\,k}}{k!} & \text{if } (i,j) \in B_2, \text{ and} \\ \alpha e^{-\lambda_3}\dfrac{\lambda_3^{\,k}}{k!} & \text{if } (i,j) \in B_3, \end{cases}$$

for $k < 0$,

$$P_{ij,k} = \begin{cases} (1-\alpha) e^{-\lambda_1}\dfrac{\lambda_1^{|k|}}{|k|!} & \text{if } (i,j) \in B_1, \\ (1-\alpha) e^{-\lambda_2}\dfrac{\lambda_2^{|k|}}{|k|!} & \text{if } (i,j) \in B_2, \text{ and} \\ (1-\alpha) e^{-\lambda_3}\dfrac{\lambda_3^{|k|}}{|k|!} & \text{if } (i,j) \in B_3, \end{cases}$$

where $\alpha \in [0]$.

New Poisson QSO Generated by 2-Partitions with A_1 is a Singleton

Let $A_1 = \{k\}$ where A_1 consists of a singleton k where $k = 0$, $k > 0$ or $k < 0$. In this paper, we will consider the cases for $k > 0$ and $k < 0$ since there are previous studies that investigated Poisson QSO generated by 2-partition with $A_1 = \{k\}$ where $k = 0$ (Ganikhodjaev & Hamzah, 2014; Hamzah & Ganikhodjaev, 2016b; Karim *et al.,* 2021b).

Case 1: For $k > 0$, then for any initial measure μ, let $\mu(A_1) = \alpha e^{-\lambda}\frac{\lambda^k}{k!}$.

$$V\mu(k)=\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}P_{ij,k}\mu(i)\mu(j)$$
$$=\sum_{i\in A_1}\sum_{j\in A_1}P_{ij,k}\mu(i)\mu(j)+\sum_{i\in A_2}\sum_{j\in A_2}P_{ij,k}\mu(i)\mu(j)+\sum_{i\in A_1}\sum_{j\in A_2}P_{ij,k}\mu(i)\mu(j)+\sum_{i\in A_2}\sum_{j\in A_1}P_{ij,k}\mu(i)\mu(j)$$
$$=\alpha e^{-\lambda_1}\frac{\lambda_1^{\,k}}{k!}[\mu(A_1)]^2+\alpha e^{-\lambda_2}\frac{\lambda_2^{\,k}}{k!}[\mu(A_2)]^2+2\alpha e^{-\lambda_3}\frac{\lambda_3^{\,k}}{k!}[\mu(A_1)\mu(A_2)]$$

$$V^2\mu(k)=\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}P_{ij,k}V\mu(i)V\mu(j)$$
$$=\sum_{i\in A_1}\sum_{j\in A_1}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i\in A_2}\sum_{j\in A_2}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i\in A_1}\sum_{j\in A_2}P_{ij,k}V\mu(i)V\mu(j)$$
$$+\sum_{i\in A_2}\sum_{j\in A_1}P_{ij,k}V\mu(i)V\mu(j)$$
$$=\alpha e^{-\lambda_1}\frac{\lambda_1^{\,k}}{k!}[V\mu(A_1)]^2+\alpha e^{-\lambda_2}\frac{\lambda_2^{\,k}}{k!}[V\mu(A_2)]^2+2\alpha e^{-\lambda_3}\frac{\lambda_3^{\,k}}{k!}[V\mu(A_1)V\mu(A_2)]$$

By using mathematical induction on the sequence $V^n\mu(k)$, we get the following recurrent equation;

$$V^{n+1}\mu(k)=\alpha e^{-\lambda_1}\frac{\lambda_1^{\,k}}{k!}[V^n\mu(A_1)]^2+\alpha e^{-\lambda_2}\frac{\lambda_2^{\,k}}{k!}[V^n\mu(A_2)]^2$$
$$+2\alpha e^{-\lambda_3}\frac{\lambda_3^{\,k}}{k!}[V^n\mu(A_1)V^n\mu(A_2)] \quad (3)$$

where n = 0, 1, 2, ... We can observe the limit behaviour of the recurrent equation in (3) is fully determined by the limit behaviour of $V^n\mu(A_1)$ and $V^n\mu(A_2)$ such that;

$$V^{n+1}\mu(A_1)=\alpha e^{-\lambda_1}\frac{\lambda_1^{\,k}}{k!}[V^n\mu(A_1)]^2+\alpha e^{-\lambda_2}\frac{\lambda_2^{\,k}}{k!}[V^n\mu(A_2)]^2$$
$$+2\alpha e^{-\lambda_3}\frac{\lambda_3^{\,k}}{k!}[V^n\mu(A_1)V^n\mu(A_2)],$$
$$V^{n+1}\mu(A_2)=(1-\alpha e^{-\lambda_1}\frac{\lambda_1^{\,k}}{k!})[V^n\mu(A_1)]^2+(1-\alpha e^{-\lambda_2}\frac{\lambda_2^{\,k}}{k!})[V^n\mu(A_2)]^2$$
$$+2(1-\alpha e^{-\lambda_3}\frac{\lambda_3^{\,k}}{k!})[V^n\mu(A_1)V^n\mu(A_2)]. \quad (4)$$

l measure μ, let $\mu(A_1)=(1-\alpha)e^{-\lambda}\frac{\lambda^{|k|}}{|k|!}$.

$$V\mu(k)=\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}P_{ij,k}\mu(i)\mu(j)$$
$$=\sum_{i\in A_1}\sum_{j\in A_1}P_{ij,k}\mu(i)\mu(j)+\sum_{i\in A_2}\sum_{j\in A_2}P_{ij,k}\mu(i)\mu(j)+\sum_{i\in A_1}\sum_{j\in A_2}P_{ij,k}\mu(i)\mu(j)+\sum_{i\in A_2}\sum_{j\in A_1}P_{ij,k}\mu(i)\mu(j)$$
$$=(1-\alpha)e^{-\lambda_1}\frac{\lambda_1^{\,|k|}}{|k|!}[\mu(A_1)]^2+(1-\alpha)e^{-\lambda_2}\frac{\lambda_2^{\,|k|}}{|k|!}[\mu(A_2)]^2+2(1-\alpha)e^{-\lambda_3}\frac{\lambda_3^{\,|k|}}{|k|!}[\mu(A_1)\mu(A_2)]$$

$$
\begin{aligned}
V^2\mu(k) &= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} P_{ij,k} V\mu(i)V\mu(j) \\
&= \sum_{i\in A_1}\sum_{j\in A_1} P_{ij,k} V\mu(i)V\mu(j) + \sum_{i\in A_2}\sum_{j\in A_2} P_{ij,k} V\mu(i)V\mu(j) + \sum_{i\in A_1}\sum_{j\in A_2} P_{ij,k} V\mu(i)V\mu(j) \\
&\quad + \sum_{i\in A_2}\sum_{j\in A_1} P_{ij,k} V\mu(i)V\mu(j) \\
&= (1-\alpha)e^{-\lambda_1}\frac{\lambda_1^{|k|}}{|k|!}[V\mu(A_1)]^2 + (1-\alpha)e^{-\lambda_2}\frac{\lambda_2^{|k|}}{|k|!}[V\mu(A_2)]^2 + 2(1-\alpha)e^{-\lambda_3}\frac{\lambda_3^{|k|}}{|k|!}[V\mu(A_1)V\mu(A_2)]
\end{aligned}
$$

By using mathematical induction on the sequence $V^n\mu(k)$, we get the following recurrent equation;

$$
\begin{aligned}
V^{n+1}\mu(k) &= (1-\alpha)e^{-\lambda_1}\frac{\lambda_1^{|k|}}{|k|!}[V^n\mu(A_1)]^2 + (1-\alpha)e^{-\lambda_2}\frac{\lambda_2^{|k|}}{|k|!}[V^n\mu(A_2)]^2 \\
&\quad +2(1-\alpha)e^{-\lambda_3}\frac{\lambda_3^{|k|}}{|k|!}[V^n\mu(A_1)V^n\mu(A_2)]
\end{aligned} \tag{5}
$$

where n = 0, 1, 2, ... We can observe the limit behaviour of the recurrent equation in (5) is fully determined by the limit behaviour of $V^n\mu(A_1)$ and $V^n\mu(A_2)$ such that;

$$
\begin{aligned}
V^{n+1}\mu(A_1) &= (1-\alpha)e^{-\lambda_1}\frac{\lambda_1^{|k|}}{|k|!}[V^n\mu(A_1)]^2 + (1-\alpha)e^{-\lambda_2}\frac{\lambda_2^{|k|}}{|k|!}[V^n\mu(A_2)]^2 \\
&\quad +2(1-\alpha)e^{-\lambda_3}\frac{\lambda_3^{|k|}}{|k|!}[V^n\mu(A_1)V^n\mu(A_2)], \\
V^{n+1}\mu(A_2) &= \left(1-(1-\alpha)e^{-\lambda_1}\frac{\lambda_1^{|k|}}{|k|!}\right)[V^n\mu(A_1)]^2 + \left(1-(1-\alpha)e^{-\lambda_2}\frac{\lambda_2^{|k|}}{|k|!}\right) \\
&\quad [V^n\mu(A_2)]^2 + 2\left(1-(1-\alpha)e^{-\lambda_3}\frac{\lambda_3^{|k|}}{|k|!}\right)[V^n\mu(A_1)V^n\mu(A_2)].
\end{aligned} \tag{6}
$$

Results and Discussion

Regularity of the Defined Poisson QSO

The recurrent equations (4) and (6) can be written as follows:

$$
W:\begin{cases} x' = A_1(\lambda_1)[x^2] + A_1(\lambda_2)[y^2] + 2A_1(\lambda_3)[xy], \\ y' = A_2(\lambda_1)[x^2] + A_2(\lambda_2)[y^2] + 2A_2(\lambda_3)[xy], \end{cases} \tag{7}
$$

where $x + y = 1$. Then, it is sufficient to find all solutions for x of x' where $0 < x < 1$. By substituting $y = 1 - x$ into x', we get the following equation concerning x,

$$
x' = A_1(\lambda_1)x^2 + A_1(\lambda_2)(1-x)^2 + 2A_1(\lambda_3)x(1-x). \tag{8}
$$

Simplifying the above equation, we get

$$
x' = \left[A_1(\lambda_1) + A_1(\lambda_2) - 2A_1(\lambda_3)\right]x^2 + 2\left[A_1(\lambda_3) - A_1(\lambda_2)\right]x + A_1(\lambda_2). \tag{9}
$$

Let $f(x) = x'$, then we have

$$
f(x) = \left[A_1(\lambda_1) + A_1(\lambda_2) - 2A_1(\lambda_3)\right]x^2 + 2\left[A_1(\lambda_3) - A_1(\lambda_2)\right]x + A_1(\lambda_2). \tag{10}
$$

By assuming $a = A(\lambda_1)$, $b = A(\lambda_2)$ and $c = A(\lambda_3)$, we can rewrite the recurrent equation above as follows:

$$f(x) = (a+b-2c)x^2 + 2(c-b)x + b, \quad (11)$$

with $0 < a, b, c < 1$.

To find the fixed point, we let $f(x) = x$ and we will get;

$$(a+b-2c)x^2 + [2(c-b)-1]x + b = 0. \quad (12)$$

The discriminant of the quadratic equation above is in the form of;

$$\Delta = 4(1-a)b + (1-2c)^2. \quad (13)$$

Suppose that the derivative $f'(x)$ is continuous and let x^* be a fixed point of equation (12), where;

$$x^* = \frac{2(b-c)+1-\sqrt{\Delta}}{2(a+b-2c)}, \quad (14)$$

and

$$f'(x) = 2(a+b-2c)x + 2(c-b). \quad (15)$$

By substituting (14) into (15), we will get;

$$f'(x^*) = 1-\sqrt{\Delta}. \quad (16)$$

Theorem 1 (Lyubich, 1992): If $0 < \Delta < 4$, then a one-dimensional QSO is regular and if $4 < \Delta < 5$, then there exists a cycle of second order and all trajectories tend to this cycle except the stationary trajectory starting with fixed point.

Theorem 2 (Lyubich, 1992): For the quadratic equation $(a + b - 2c)x^2 + 2(c - b)x + b = x$, where $0 < a, b, c < 1$ and its discriminant, Δ, one has that $4 < \Delta < 5$ if and only if

$a \in \left[0, \frac{1}{4}\right], b > \frac{3}{4(1-a)}$ and $c < \frac{1}{2} - \sqrt{1-(1-a)b}$ or $c > \frac{1}{2} + \sqrt{1-(1-a)b}$.

Examples

In this part, we will demonstrate the existence of a fixed point in the system of equations in (7) by presenting several applications of Theorem 1 on the new class of Poisson QSO generated by 2-partition defined on the set of all integers. We recall the system of equations is given by

$$W: \begin{cases} x' = A_1(\lambda_1)[x^2] + A_1(\lambda_2)[y^2] + 2A_1(\lambda_3)[xy], \\ y' = A_2(\lambda_1)[x^2] + A_2(\lambda_2)[y^2] + 2A_2(\lambda_3)[xy]. \end{cases}$$

where $a = A(\lambda_1)$, $b = A(\lambda_2)$ and $c = A(\lambda_3)$.

Example 1: Let $\alpha = 0.5$, $\lambda_1 = 1.5$, $\lambda_2 = 2.5$, $\lambda_3 = 3.5$ and $k = 1$ with $\mu(A_1) = 0.173804856326994$. We may calculate the heredity coefficients, *a, b,* and *c* after substituting and using the values of the parameters and measure given above into (7). From programming software, we have

$a = 0.167347620111322$,
$b = 0.102606248279874$,
$c = 0.0528454209890572$

where the fixed point is (0.0946575757445665, 0.905342424255434).

Figure 1 shows that the fixed points of equation (7) for Example 1 are x = 0.0946575757445665 and y = 0.905342424255434. The discriminant is $\Delta_1 = 4(1-a)b+(1-2c)^2 = 1.14153021740856$. Since $0 < \Delta_1 < 4$, hence, the fixed point (0.0946575757445665, 0.905342424255434) is an attractive fixed point and is a regular transformation since a limit exists.

Example 2: Let $\alpha = 0.5$, $\lambda_1 = 1.5$, $\lambda_2 = 1.5$, $\lambda_3 = 1.5$ and $k = -4$ with $\mu(A_1) = 0.000789753463167492$. Therefore, by programming software, the heredity coefficients, *a, b* and *c* are

$a = 0.0235332590781547$,
$b = 0.0235332590781547$,
$c = 0.0235332590781547$

where the fixed point is (0.0235332590781547, 0.976466740921845).

From Figure 2, the fixed points of equation (7) for Example 2 are $x = 0.0235332590781547$ and $y = 0.976466740921845$. The discriminant is $\Delta_3 = 4(1-a)b + (1-2c)^2 = 1.00000000000000$. Since $0 < \Delta_2 < 4$, hence, the fixed point (0.0235332590781547, 0.976466740921845) is an attractive fixed point and it is a regular transformation since there exists one limit.

Example 3: Let $\alpha = 0.75$, $\lambda_1 = 150$, $\lambda_2 = 100$, $\lambda_3 = 1.5$ and $k = 2$ with $\mu(A_1) = 0.130887665125383$.

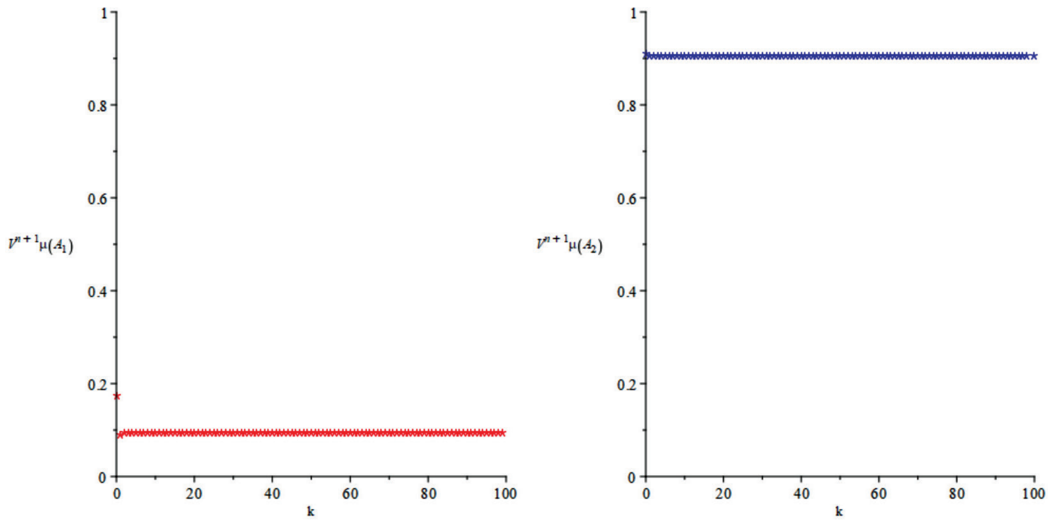

Figure 1: Plot orbit for x and y for Example 1

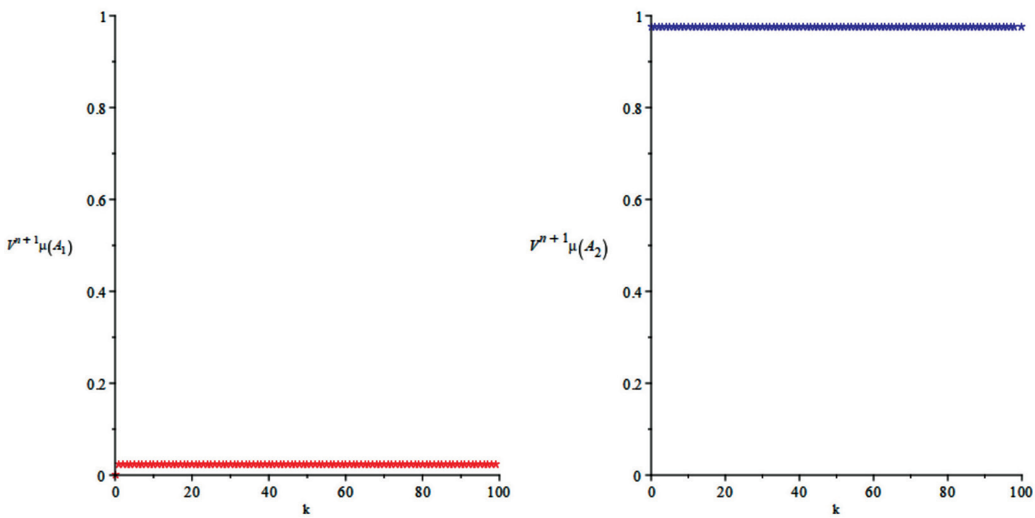

Figure 2: Plot orbit for x and y for Example 2

Therefore, the heredity coefficients, a, b, and c obtained from the programming software are

$a = 8437.50000000000e^{-150}$,
$b = 3750.00000000000e^{-100}$,
$c = 0.188266072625237$

where the fixed point is ($2.25763179326745 \times 10^{-40}$, 1).

From Figure 3, the fixed point of equation (7) from Example 3 is $x = 2.25763179326745 \times 10^{-40}$ and $y = 1$. The discriminant is $\Delta_3 = 4(1-a)b + (1-2c)^2 = 0.388712165905976$. Since $0 < \Delta_3 < 4$, hence, the fixed point ($2.25763179326745 \times 10^{-40}$, attractive fixed point and a regular transformation.

Example 4: Let $\alpha = 0.15$, $\lambda_1 = 1.5$, $\lambda_2 = 2.5$, $\lambda_3 = 5.5$ and $k = -2$ with $\mu(A_1) = 0.148339353808767$. Hence, the heredity coefficients, *a, b* and *c* are

$a = 0.213368215641937$,
$b = 0.218038277594731$,
$c = 0.157215127442446$

where the fixed point is (0.198500571026844, 0.801499428973156).

From Figure 4, the fixed points of equation (7) from Example 4 are $x = 0.198500571026844$ and $y = 0.801499428973156$. The discriminant is $\Delta_4 = 4(1 - a)b + (1 - 2c)^2 = 1.15606923286800$. Since $0 < \Delta_4 < 4$, thus, the fixed point (0.198500571026844, 0.801499428973156) is an attractive fixed point and a regular transformation since one limit exists.

Conclusions

In this paper, we construct a new class of Poisson QSO generated by 2-partition on state space of all integers. We consider a few cases where $k > 0$ and $k < 0$ when A_1 is a singleton. We show that

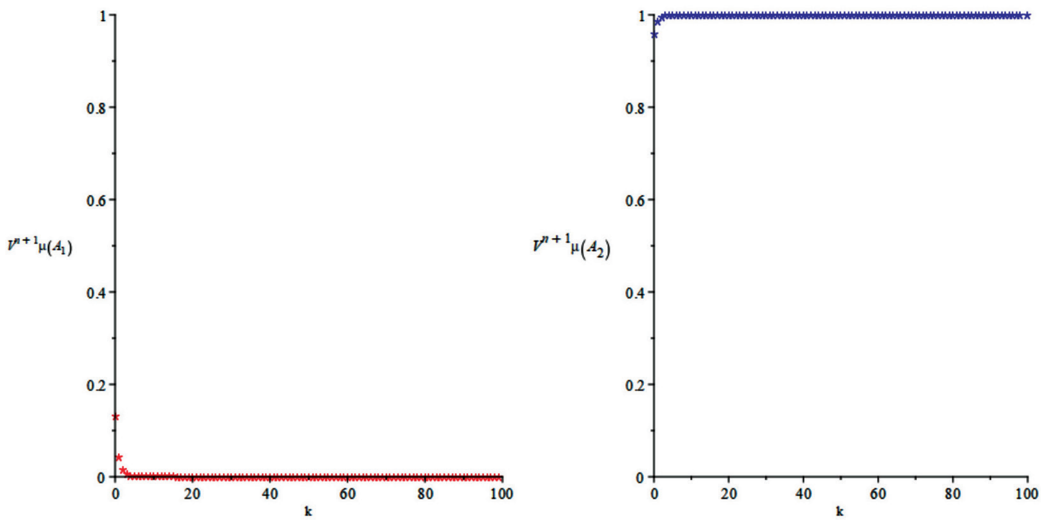

Figure 3: Plot orbit for x and y for Example 3

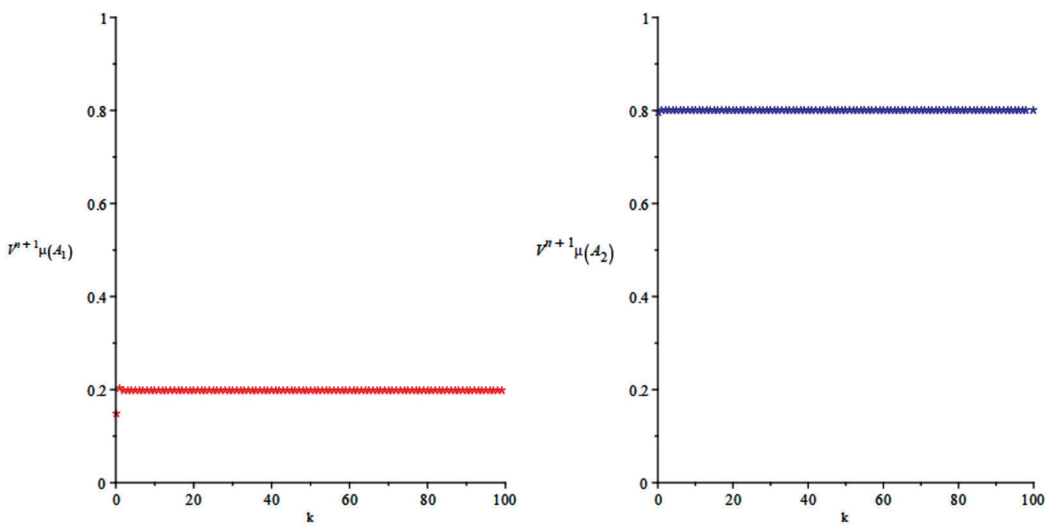

Figure 4: Plot orbit for x and y for Example 4

the operator is a regular transformation for some values of parameters. It is suggested that future QSO research on the set of all integers include more cases with more partitions and parameters.

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Conflict of Interest Statement

The authors declare that they have no conflict of interest.

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