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# ON LEBESGUE QUADRATIC STOCHASTIC OPERATORS WITH EXPONENTIAL MEASURE GENERATED BY 3-PARTITION

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**Abstract:** Quadratic Stochastic Operator (QSO) is a continuously expanding topic in nonlinear operator theory due to its immense applications in various disciplines. Inspired by the notion of infinite state space, as there is limited literature on the QSO study defined on such a state space, we consider a QSO class on continuous state space in this work. It is known as Lebesgue QSO, which is an exponential measure generated by three measurable partitions with three parameters. We specify two distinct cases of three parameters, which are represented by reducible QSOs. We demonstrate that such a reducible QSO can be reduced to a one-dimensional simplex. Consequently, we analyse the dynamics of such operators by employing the first derivative method and show that the operators may have either an attracting fixed point to indicate the existence of a strong limit or a non-attracting fixed point to suggest the presence of a second-order cycle. Corresponding to a strong limit of the sequence of the reduced QSO, such an operator is regular. Meanwhile, such an operator is a nonregular transformation when a second-order cycle exists.

Keywords: Quadratic stochastic operator, Lebesgue measure, dynamics, regularity.

## Introduction

Sustainable development has been an essential topic in diverse fields due to the demands and challenges in the sustainable management practice, where mathematical approaches are acknowledged to play huge roles in sustainable development in arriving at the understanding, prediction, and control of the development process. It is recognised that mathematical and numerical models, which rely on dynamical systems in control theory are essential tools for helping us evaluate sustainability paradoxes in a complex world with various dimensions, values, and scales. A nonlinear operator, which refers to a complex system model may describe a dynamical system. The simplest nonlinear operator, the Quadratic Stochastic Operator (QSO) has been recognised as a significant analysis source of dynamical modelling and properties study in many fields including biology, physics, chemistry, and economics.

Bernstein's attempt to develop the mathematical framework for the study of the connection between Galton's regression law and Mendel's crossing law led to the origination of the notion of a QSO in the early 1920s (Bernstein, 1922; 1942). The importance of QSO as a useful source of dynamical modelling and properties analysis has been immensely acknowledged across diverse fields, notably in population genetics and evolutionary games (Volterra, 1927; Hofbauer *et al*., 1982; Nagylaki, 1983a; 1983b; Losert & Akin, 1983; Akin & Losert, 1984; Hofbauer & Sigmund, 1998).

Initially, Bernstein (1942) introduced the QSO system in relation to population genetics. The QSO can be interpreted as follows when it is used in biology to depict the time evolution of diverse species.

Assume that $m$ species make up a population. Let $x^{(0)} = \left(x_1^{(0)}, \ldots, x_m^{(0)}\right)$ be an initial state of a probability distribution of a species and $P_{ij,k}$ be a probability for which $i^{th}$ and $j^{th}$ species crossbreed to generate $k^{th}$ species. Following that, a total probability can be employed to represent a probability distribution $x^{(1)} = \left(x_1^{(1)},...,x_m^{(1)}\right)$ of the species in the first generation, as illustrated below:

$$x_k^{(1)} = \sum_{i,j=1}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)},\ k = 1,...,m.$$

As a result, the relationship $x^{(0)} \to x^{(1)}$ is a mapping $V$ also referred to as an evolutionary operator, where the population evolves to the first generation $x^{(1)} = V\left(x^{(0)}\right)$ from an initial state $x^{(0)}$, followed by the second generation $x^{(2)} = V\left(x^{(1)}\right) = V\left(V\left(x^{(0)}\right)\right) = V^{(2)}\left(x^{(0)}\right)$, and so on. Consequently, a discrete dynamical system can be utilised to express the evolution states of the population system as follows:

$$x^{(0)}, x^{(1)} = V\left(x^{(0)}\right), x^{(2)} = V^{(2)}\left(x^{(0)}\right),...$$

This demonstrates how a QSO can define the next generation distribution given the present generation distribution.

Given that the applications of QSO significantly contribute to understanding various physical and natural phenomena, numerous researchers have dedicated their study to presenting different classes of QSO and investigating their behaviour. Recently, diverse QSO classes on infinite state space have been studied, i.e., Geometric QSO (Ganikhodjaev & Hamzah, 2015a; Karim *et al.*, 2019; 2020; Khaled & Hee, 2021), Poisson QSO (Ganikhodjaev & Hamzah, 2014; 2016), Gaussian QSO (Ganikhodjaev & Hamzah, 2015c; 2017; 2018), Lebesgue QSO (Ganikhodjaev & Hamzah, 2015b; Ganikhodjaev *et al.*, 2017), and QSO generated by mixture distributions (Ganikhodjaev & Khaled, 2021).

Ganikhodjaev (2016) proposed that the QSO analysis on infinite state space could be conducted through a finite-dimensional setting analysis of such operators. Given these findings, Karim *et al.* (2019; 2020) consider Geometric QSO generated by 2-partition, where each partition is given a unique parameter. Hence such a QSO is reduced to a $(m - 1)$-dimensional simplex for $m = 2$. As a result, the reduced QSO analysis shall describe the study of the asymptotic behaviour of nonlinear operators.

On the other hand, Hamzah *et al.* (2022) introduced a class of QSO named Lebesgue QSO with exponential measure generated by 3-partition, where a reduced QSO on a two-dimensional simplex is considered as shown below:

$$W_1: \begin{cases} x_1' = a_{11}\left(x_1^2 + 2x_1x_3 + x_3^2\right) + a_{22}x_2^2 + 2a_{12}\left(x_1x_2 + x_2x_3\right), \\ x_2' = b_{11}\left(x_1^2 + 2x_1x_3 + x_3^2\right) + b_{22}x_2^2 + 2b_{12}\left(x_1x_2 + x_2x_3\right), \\ x_3' = c_{11}\left(x_1^2 + 2x_1x_3 + x_3^2\right) + c_{22}x_2^2 + 2c_{12}\left(x_1x_2 + x_2x_3\right). \end{cases} \quad (1)$$

For this particular case, the QSO can be further reduced to a one-dimensional simplex and is derived from one of three cases of 3-partition with three different parameters, where $\mu_{11} = \mu_{33} = \mu_{13} \neq \mu_{12} = \mu_{23} \neq \mu_{22}$. Therefore, in this current research, we would like to study the dynamics of such a QSO for another two cases, where $\mu_{11} = \mu_{22} = \mu_{12} \neq \mu_{13} = \mu_{23} \neq \mu_{33}$ and $\mu_{22} = \mu_{33} = \mu_{23} \neq \mu_{12} = \mu_{13} \neq \mu_{11}$, which have yet to be studied.

The order of the article is as follows: The required definitions, ideas, and theorems are provided in the following section. The results and analysis of the 3-partition Lebesgue QSO with three distinct parameters in construction, dynamics, and regularity are then presented.

## Methodologies and Preliminaries

Consider that $X$ is a state space and $\mathcal{F}$ is a $\sigma$-algebra of subsets of $X$ for which, $(X,\mathcal{F})$ and $S(X,\mathcal{F})$ as a measurable space and an all probability measures set on such a measurable space, respectively. A family of functions $\{P(x,y,A): x,y \in X, A \in \mathcal{F}\}$ on with the following $X \times X \times \mathcal{F}$ conditions is then defined.

(i) $P(x,y,.) \in S(X,\mathcal{F})$ is a probability measure, where $P(x,y,.)$: $\mathcal{F} \to [0,1]$, for $\forall x,y \in X$,

(ii) $P(x,y,A)$ is a function of two variables, $x$ and $y$ with a fixed $A \in \mathcal{F}$, which is measurable on $(X \times X, \mathcal{F} \otimes \mathcal{F})$, and

(iii) $P(x,y,A) = P(y,x,A)$.

A QSO, $V : S(X,\mathcal{F}) \rightarrow S(X,\mathcal{F})$ is described as follows:

$$V\lambda(A) = \int_X \int_X P(x,y,A)\, d\lambda(x)\, d\lambda(y), \quad (2)$$

where $\lambda \in S(X,\mathcal{F})$ is an initial measure arbitrarily. For a finite state space $X$, where $X = \{1,...,m\}$ and $P(X)$ is the analogous $\sigma$-algebra on $X$, the probability measures set $S(X,\mathcal{F})$ is known as an $(m - 1)$-dimensional simplex defined by:

$$S^{m-1} = \left( \mathbf{x} = (x_1, x_2, ..., x_m) \in \sim^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \right). \quad (3)$$

**Definition 1:** A mapping $V : S^{m-1} \rightarrow S^{m-1}$ with:

$$(Vx)_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad (4)$$

is called a QSO, where $P_{ij,k} \geq 0$, $P_{ij,k} = P_{ji,k}$, and $\sum_{k=1}^{m} P_{ij,k} = 1$ *for* $i,j,k \in \{1,...,m\}$.

Suppose that $\xi$ be an $m$-measurable partition of $X$, where $\xi = \{A_1,...,A_m\}$, $A_i \subset X$, and $A_i \cap A_j = \emptyset$ for any $i, j \in \{1,...,m\}$. Now, consider a partition $\zeta = \left\{ B_{ij} = \bigcup_{i,j=1}^{m} (A_i \times A_j) : B_{ij} = B_{ji} \right\}$ on $X \times X$, which corresponds to $\xi$. By considering such a partition on $X$, one may specify the probability measure $P(x,y,A)$ as follows:

$$P(x,y,A) = \mu_{ij}(A), \quad (5)$$

for $(x,y) \in B_{ij}$. For any $\lambda \in S(X,\mathcal{F})$ and $A \in \mathcal{F}$, one shall attain:

$$V\lambda(A) = \int_X \int_X P(x,y,A)\, d\lambda(x)\, d\lambda(y)$$
$$= \sum_{i,j=1}^{m} \mu_{ij}(A) \lambda(A_i) \lambda(A_j).$$

Since $\lambda$ is an initial measure with $V^0\lambda = \lambda$ and $\{V^n\lambda: n = 0,1,...\}$ is the initial measure $\lambda$ trajectory behaviour, where $V^{n+1}\lambda = V(V^n\lambda)$, then, one may acquire the following recurrent equation:

$$V^{n+1}\lambda(A) = \sum_{i,j=1}^{m} \mu_{ij}(A) V^n\lambda(A_i) V^n\lambda(A_j),$$

with

$$V^{n+1}\lambda(A_k) = \sum_{i,j=1}^{m} \mu_{ij}(A_k) V^n\lambda(A_i) V^n\lambda(A_j), \quad (6)$$

where $A_k \subset X$ for $k = 1,...,m$.

Let $x_k^{(n)} = V^n\lambda(A_k)$ and $P_{ij,k} = \mu_{ji}(A_k)$. Then, for $x^{(n)} \in S^{m-1}$, the recurrent equation in (6) may be expressed as follows:

$$(Wx)_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad (7)$$

for all $k = 1,...,m$.

It is shown that for fixed measurable finite partitions on the infinite state space $X$ and a probability measures family in (5), the QSO $V$ in (6) can be estimated by a QSO $W$ in (7). In this study, we consider a class of Lebesgue QSO, which can be described as follows:

**Definition 2:** Suppose that $X = [0,1]$ and $F$ is a Borel $\sigma$-algebra on $[0,1]$, a transformation $V$ in (2) is known as a Lebesgue QSO.

Consider a continuous set $X = [0,1]$ and a $\sigma$-algebra $\mathcal{F}$ on $X$. We choose a probability measure $\mu_q$ with the following form:

$$\mu_q(A) = \frac{\ln(q+1)}{q} \int_A (q+1)^x\, dx, \quad (8)$$

for an arbitrary $q \in \bullet$, where $\bullet$ is a set of natural numbers. A QSO defined by such a probability measure is named Lebesgue QSO with exponential measure.

The construction, dynamics, and regularity of the Lebesgue QSO with exponential measure generated by 3-partition with three parameters will all be covered in the following section.

## Results and Discussion

### *Lebesgue QSO with Exponential Measure Generated by 3-Partition with Three Parameters*

Let $\xi = \{A_1,A_2,A_3\}$ be a measurable 3-partition of the continuous set $X = [0,1]$ and $\zeta = \{B_{11},B_{22},B_{33}, B_{12},B_{13},B_{23}\}$ be a partition of the Cartesian product $X \times X = [0,1] \times [0,1]$, where $B_{ii} = A_i \times A_i$ and $B_{ij} = (A_i \times A_j) \cup (A_j \times A_i)$ for $i \neq j$ and $i,j = 1,2,3$. Consequently, determine a family $\{\mu_{q_{ij}}: i,j = 1,2,3$ and $q_{ij} \in \bullet\}$ of the measure as shown in (8) with parameters, $q_{11} = q_1$, $q_{22} = q_2$, $q_{33} = q_3$, $q_{12} = q_4$, $q_{13} = q_5$, and $q_{23} = q_6$. Subsequently, the probability measure, $P(x,y,A)$ is defined as follows:

$$P(x,y,A)=\mu_{q_{ij}}(A) \text{ if } (x,y)\in B_{ij},\ i,j=1,2,3, \quad (9)$$

for arbitrary $A = [a,b] \in \mathcal{F}$. Now, let any continuous initial measure $\lambda \in S(X,\mathcal{F})$ be a probability measure as defined in (9). Correspondingly, we define $A = [\alpha,\beta] \subset [0,1]$ and $\int_{A_k} d\lambda = \lambda(A_k)$ for $k = 1,2,3$, where $A_1 = [0,\alpha_1]$, $A_2 = [\alpha_1,\alpha_2]$, and $A_3 = [\alpha_2,1]$. Note that $\alpha_1 < \alpha_2$ and $\alpha_1,\alpha_2 \in (0,1)$. We may construct such a Lebesgue QSO as follows:

$$\begin{aligned}
V(\lambda(A)) = \lambda'(A) &= \int_0^1\int_0^1 P(x,y,A)\,d\lambda(x)\,d\lambda(y)\\
&= \int_{A_1}\int_{A_1}\mu_{q_1}(A)\,d\lambda(x)\,d\lambda(y)+\int_{A_2}\int_{A_2}\mu_{q_2}(A)\,d\lambda(x)\,d\lambda(y)+\int_{A_3}\int_{A_3}\mu_{q_3}(A)\,d\lambda(x)\,d\lambda(y)\\
&+\int_{A_1}\int_{A_2}\mu_{q_4}(A)\,d\lambda(x)\,d\lambda(y)+\int_{A_2}\int_{A_1}\mu_{q_4}(A)\,d\lambda(x)\,d\lambda(y)+\int_{A_1}\int_{A_3}\mu_{q_5}(A)\,d\lambda(x)\,d\lambda(y)\\
&+\int_{A_3}\int_{A_1}\mu_{q_5}(A)\,d\lambda(x)\,d\lambda(y)+\int_{A_2}\int_{A_3}\mu_{q_6}(A)\,d\lambda(x)\,d\lambda(y)+\int_{A_3}\int_{A_2}\mu_{q_6}(A)\,d\lambda(x)\,d\lambda(y)\\
&= \mu_{q_1}(A)\left[\lambda^2(A_1)\right]+\mu_{q_2}(A)\left[\lambda^2(A_2)\right]+\mu_{q_3}(A)\left[\lambda^2(A_3)\right]+2\mu_{q_4}(A)\left[2\lambda(A_1)\lambda(A_2)\right]\\
&+2\mu_{q_5}(A)\left[2\lambda(A_1)\lambda(A_3)\right]+2\mu_{q_6}(A)\left[2\lambda(A_2)\lambda(A_3)\right],
\end{aligned}$$

and

$$\begin{aligned}
V^{n+1}(\lambda(A)) &= \mu_{q_1}(A)\left(V^n\lambda(A_1)\right)^2+\mu_{q_2}(A)\left(V^n\lambda(A_2)\right)^2\\
&+\mu_{q_3}(A)\left(V^n\lambda(A_3)\right)^2+2\mu_{q_4}(A)V^n\lambda(A_1)V^n\lambda(A_2)\\
&+2\mu_{q_5}(A)V^n\lambda(A_1)V^n\lambda(A_3)+2\mu_{q_6}(A)V^n\lambda(A_2)V^n\lambda(A_3),
\end{aligned}$$

with

$$\begin{aligned}
V^{n+1}\lambda(A_k) &= \mu_{q_1}(A_k)\left(V^n\lambda(A_1)\right)^2+\mu_{q_2}(A_k)\left(V^n\lambda(A_2)\right)^2\\
&+\mu_{q_3}(A_k)\left(V^n\lambda(A_3)\right)^2+2\mu_{q_4}(A_k)V^n\lambda(A_1)V^n\lambda(A_2)\\
&+2\mu_{q_5}(A_k)V^n\lambda(A_1)V^n\lambda(A_3)+2\mu_{q_6}(A_k)V^n\lambda(A_2)V^n\lambda(A_3),
\end{aligned}\quad (10)$$

where $n = 0,1,\ldots$ and $k = 1,2,3$.

Based on the recurrent equations in (10), it is evident that such equations can be rewritten as a two-dimensional QSO with the following form:

$$\begin{aligned}
(Wx)_1 &= a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{12}x_1x_2+2a_{13}x_1x_3+2a_{23}x_2x_3,\\
(Wx)_2 &= b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+2b_{12}x_1x_2+2b_{13}x_1x_3+2b_{23}x_2x_3,\\
(Wx)_3 &= c_{11}x_1^2+c_{22}x_2^2+c_{33}x_3^2+2c_{12}x_1x_2+2c_{13}x_1x_3+2c_{23}x_2x_3,
\end{aligned}\quad (11)$$

where $a_{ij} = P_{ij,1}$, $b_{ij} = P_{ij,2}$, and $c_{ij} = P_{ij,3}$ for $i,j = 1,2,3$ and $a_{ij}+b_{ij}+c_{ij} = 1$.

As mentioned earlier, throughout this article, we will consider two cases where $\mu_{q11} = \mu_{q22} = \mu_{q12} \neq \mu_{q13} = \mu_{q23} \neq \mu_{q33}$ and $\mu_{q22} = \mu_{q33} = \mu_{q23} \neq \mu_{q12} = \mu_{q13} \neq \mu_{q11}$, producing the following system of equations, respectively:

$$W_2 : \begin{cases} x_1' = a_{22}\left(x_1^2 + 2x_1x_2 + x_2^2\right) + a_{33}x_3^2 + 2a_{23}\left(x_1x_3 + x_2x_3\right), \\ x_2' = b_{22}\left(x_1^2 + 2x_1x_2 + x_2^2\right) + b_{33}x_3^2 + 2b_{23}\left(x_1x_3 + x_2x_3\right), \\ x_3' = c_{22}\left(x_1^2 + 2x_1x_2 + x_2^2\right) + c_{33}x_3^2 + 2c_{23}\left(x_1x_3 + x_2x_3\right). \end{cases} \quad (12)$$

and

$$W_3 : \begin{cases} x_1' = a_{33}\left(x_2^2 + 2x_2x_3 + x_3^2\right) + a_{11}x_1^2 + 2a_{13}\left(x_1x_2 + x_1x_3\right), \\ x_2' = b_{33}\left(x_2^2 + 2x_2x_3 + x_3^2\right) + b_{11}x_1^2 + 2b_{13}\left(x_1x_2 + x_1x_3\right), \\ x_3' = c_{33}\left(x_2^2 + 2x_2x_3 + x_3^2\right) + c_{11}x_1^2 + 2c_{13}\left(x_1x_2 + x_1x_3\right). \end{cases} \quad (13)$$

We shall discuss the trajectory behaviour of both QSO $W_2$ and $W_3$ in the succeeding section.

### *Dynamics of Lebesgue QSO with Exponential Measure Generated by 3-Partition with Three Parameters*

The dynamics of the 3-partition of Lebesgue QSO represented by the system of equations in (12) and (13) can be described by their fixed points and periodic points.

**Remark 1:** Suppose that $x^*$ is a hyperbolic fixed point, where $|f'(x^*)| \neq 1$ and $f'(x)$ be the first derivative of a function $f(x)$. Then, the following statements hold:

(i) If $|f'(x^*)| < 1$, then, $x^*$ is an attracting hyperbolic fixed point.

(ii) If $|f'(x^*)| > 1$, then, $x^*$ is a repelling hyperbolic fixed point.

**Remark 2:** (Lyubich, 1974) Given a one-dimensional QSO $W$:

$$W : \begin{cases} x_1' = ax_1^2 + 2bx_1x_2 + cx_2^2 \\ x_2' = (1-a)x_1^2 + 2(1-b)x_1x_2 + (1-c)x_2^2 \end{cases}, \quad (14)$$

with a function:

$$f(x_1) = (a - 2b + c)x_1^2 + 2(b - c)x_1 + c, \quad (15)$$

for $0 < a,b,c < 1$, $x_1 + x_2 = 1$ and $\Delta = 4(1 - a)c + (1 - 2b)^2$ is the discriminant of such a function, the following holds:

(i) The fixed point $x_1^* \in (0,1)$ has the following form:

$$x_1^* = \frac{-2(b-c) + 1 - \sqrt{\Delta}}{2(a - 2b + c)}.$$

(ii) If $0 < \Delta < 4$, then, $x_1^*$ is an attractive fixed point.

(iii) If $4 < \Delta < 5$, then, $x_1^*$ is a repelling fixed point.

**Theorem 1:** (Lyubich, 1974) If $0 < \Delta < 4$, consequently, every trajectory will converge to a fixed point.

**Theorem 2:** (Lyubich, 1974) If $4 < \Delta < 5$, correspondingly, there exists a second-order cycle, where all trajectories incline to this cycle except for the stationary point starting with the fixed point.

The above remarks and theorems are implied to describe the dynamics of a QSO in a one-dimensional setting studied in Lyubich (1974). Be mindful that in this article, we consider a QSO in a two-dimensional simplex, which can be further reduced to a one-dimensional setting as represented by the QSO $W$ in (14).

By referring to Remark 2, we may obtain either an attracting fixed point or a repelling fixed point. However, in the case of the QSO in a two-dimensional simplex, we shall infer the repelling fixed point as a non-attracting fixed point. Due to the vagueness of the characteristics of the fixed point, which is not attractive enough to be saddled or repelled, we shall imply it as a non-attracting fixed point in the following discussion.

In Hamzah *et al.* (2022), the dynamics of such a Lebesgue QSO of a particular case, where $\mu_{q11} = \mu_{q33} = \mu_{q13} \neq \mu_{q12} = \mu_{q23} \neq \mu_{q22}$ has been presented comprehensively. Such a QSO is reduced to a one-dimensional simplex by substituting the summation of and as one variable and as another variable, which eventually takes the form of the QSO in (14).

Let us define the following constants:

$$a_{11} = \frac{\ln(q_1+1)}{q_1}\int_{A_1}(q_1+1)^x\,dx,\ a_{22} = \frac{\ln(q_2+1)}{q_2}\int_{A_1}(q_2+1)^x\,dx,\ a_{33} = \frac{\ln(q_3+1)}{q_3}\int_{A_1}(q_3+1)^x\,dx,$$

$$a_{12} = \frac{\ln(q_4+1)}{q_4}\int_{A_1}(q_4+1)^x\,dx,\ a_{13} = \frac{\ln(q_5+1)}{q_5}\int_{A_1}(q_5+1)^x\,dx,\ a_{23} = \frac{\ln(q_6+1)}{q_6}\int_{A_1}(q_6+1)^x\,dx,$$

$$b_{11} = \frac{\ln(q_1+1)}{q_1}\int_{A_2}(q_1+1)^x\,dx,\ b_{22} = \frac{\ln(q_2+1)}{q_2}\int_{A_2}(q_2+1)^x\,dx,\ b_{33} = \frac{\ln(q_3+1)}{q_3}\int_{A_2}(q_3+1)^x\,dx,$$

$$b_{12} = \frac{\ln(q_4+1)}{q_4}\int_{A_2}(q_4+1)^x\,dx,\ b_{13} = \frac{\ln(q_5+1)}{q_5}\int_{A_2}(q_5+1)^x\,dx,\ b_{23} = \frac{\ln(q_6+1)}{q_6}\int_{A_2}(q_6+1)^x\,dx,$$

$$c_{11} = \frac{\ln(q_1+1)}{q_1}\int_{A_1}(q_1+1)^x\,dx,\ c_{22} = \frac{\ln(q_2+1)}{q_2}\int_{A_3}(q_2+1)^x\,dx,\ c_{33} = \frac{\ln(q_3+1)}{q_3}\int_{A_3}(q_3+1)^x\,dx,$$

$$c_{12} = \frac{\ln(q_4+1)}{q_4}\int_{A_1}(q_4+1)^x\,dx,\ c_{13} = \frac{\ln(q_5+1)}{q_5}\int_{A_3}(q_5+1)^x\,dx,\ c_{23} = \frac{\ln(q_6+1)}{q_6}\int_{A_3}(q_6+1)^x\,dx.$$

**Remark 3:** (Hamzah *et al.*, 2022) Let $\Delta_v = 4(1 - b_{22})b_{11} + (1 - 2b_{12})^2$. For $W_1$ given by (1), we have:

(i)

$$Fix(W_1) = \left\{\left(x_1^*, x_2^*, x_3^*\right)\right\}, \text{ where}$$

$$x_1^* = (a_{11} - 2a_{12} + a_{22})(x_2^*)^2 + 2(a_{12} - a_{11})x_2^* + a_{11},$$

$$x_2^* = \frac{-2(b_{12} - b_{11}) + 1 - \sqrt{\Delta_v}}{2(b_{11} - 2b_{12} + b_{22})},$$

$$x_3^* = (c_{11} - 2c_{12} + c_{22})(x_2^*)^2 + 2(c_{12} - c_{11})x_2^* + c_{11}.$$

(ii)

$$Per_2(W_1) = \left\{\left(x_{1,a}^*, x_{2,a}^*, x_{3,a}^*\right), \left(x_{1,b}^*, x_{2,b}^*, x_{3,b}^*\right)\right\}, \text{ where}$$

$$x_{1,a}^* = (a_{11} - 2a_{12} + a_{22})(x_{2,b}^*)^2 + 2(a_{12} - a_{11})x_{2,b}^* + a_{11},$$

$$x_{2,a}^* = \frac{-2(b_{12} - b_{11}) - 1 + \sqrt{\Delta_v - 4}}{2(b_{11} - 2b_{12} + b_{22})},$$

$$x_{3,a}^* = (c_{11} - 2c_{12} + c_{22})(x_{2,b}^*)^2 + 2(c_{12} - c_{11})x_{2,b}^* + c_{11},$$

and

$$x_{1,b}^{*} = \left(a_{11} - 2a_{12} + a_{22}\right)\left(x_{2,a}^{*}\right)^{2} + 2\left(a_{12} - a_{11}\right)x_{2,a}^{*} + a_{11},$$
$$x_{2,b}^{*} = \frac{-2\left(b_{12} - b_{11}\right) - 1 - \sqrt{\Delta_{v} - 4}}{2\left(b_{11} - 2b_{12} + b_{22}\right)},$$
$$x_{3,b}^{*} = \left(c_{11} - 2c_{12} + c_{22}\right)\left(x_{2,a}^{*}\right)^{2} + 2\left(c_{12} - c_{11}\right)x_{2,a}^{*} + c_{11}.$$

The above remark allows us to establish the following for the operator $W_2$ and $W_3$.

**Proposition 1:** Let $\Delta_v = 4(1 - c_{33})c_{22} + (1 - 2c_{23})^2$. For $W_2$ given by (12), we have:

(i) $Fix(W_2) = \left\{\left(x_1^*, x_2^*, x_3^*\right)\right\}$, where

$$x_{1}^{*} = \left(a_{22} - 2a_{23} + a_{33}\right)\left(x_{3}^{*}\right)^{2} + 2\left(a_{23} - a_{22}\right)x_{3}^{*} + a_{22},$$
$$x_{2}^{*} = \left(b_{22} - 2b_{23} + b_{33}\right)\left(x_{3}^{*}\right)^{2} + 2\left(b_{23} - b_{22}\right)x_{3}^{*} + b_{22},$$
$$x_{3}^{*} = \frac{-2\left(c_{23} - c_{22}\right) + 1 - \sqrt{\Delta_{v}}}{2\left(c_{22} - 2c_{23} + c_{33}\right)}.$$

(ii) $Per_2(W_2) = \left\{\left(x_{1,a}^*, x_{2,a}^*, x_{3,a}^*\right), \left(x_{1,b}^*, x_{2,b}^*, x_{3,b}^*\right)\right\}$, where

$$x_{1,a}^{*} = \left(a_{22} - 2a_{23} + a_{33}\right)\left(x_{3,b}^{*}\right)^{2} + 2\left(a_{23} - a_{22}\right)x_{3,b}^{*} + a_{22},$$
$$x_{2,a}^{*} = \left(b_{22} - 2b_{23} + b_{33}\right)\left(x_{3,b}^{*}\right)^{2} + 2\left(b_{23} - b_{22}\right)x_{3,b}^{*} + b_{22},$$
$$x_{3,a}^{*} = \frac{-2\left(c_{23} - c_{22}\right) - 1 + \sqrt{\Delta_{v} - 4}}{2\left(c_{22} - 2c_{23} + c_{33}\right)},$$

and

$$x_{1,b}^{*} = \left(a_{22} - 2a_{23} + a_{33}\right)\left(x_{3,a}^{*}\right)^{2} + 2\left(a_{23} - a_{22}\right)x_{3,a}^{*} + a_{22},$$
$$x_{2,b}^{*} = \left(b_{22} - 2b_{23} + b_{33}\right)\left(x_{3,a}^{*}\right)^{2} + 2\left(b_{23} - b_{22}\right)x_{3,a}^{*} + b_{22},$$
$$x_{3,b}^{*} = \frac{-2\left(c_{23} - c_{22}\right) - 1 - \sqrt{\Delta_{v} - 4}}{2\left(c_{22} - 2c_{23} + c_{33}\right)}.$$

*Proof*

(i) Let $u = x_1 + x_2$ and $v = x_3$. Then, we may reduce such an operator to a one-dimensional simplex, where

$$W_{\alpha_2} : \begin{cases} u' = \left(a_{22} + b_{22}\right)u^2 + \left(a_{33} + b_{33}\right)v^2 + 2\left(a_{23} + b_{23}\right)uv, \\ v' = c_{22}u^2 + c_{33}v^2 + 2c_{23}uv. \end{cases} \tag{16}$$

Since $u + v = 1$, it is easy to solve for $v$, where we let the following:

$$\begin{aligned} f(v) &= c_{22}(1-v)^2 + 2c_{23}(1-v)v + c_{33}v^2 \\ &= (c_{22} - 2c_{23} + c_{33})v^2 + 2(c_{23} - c_{22})v + c_{22}. \end{aligned} \tag{17}$$

Then, for $f(v) = v$, the solutions are as follows:

$$v = \frac{-2(c_{23} - c_{22}) + 1 \pm \sqrt{\Delta_v}}{2(c_{22} - 2c_{23} + c_{33})}, \tag{18}$$

for $\Delta_v = 4(1 - c_{33})c_{22} + (1 - 2c_{23})^2$. Due to the fact that $0 < \Delta_v < 5$ and $\sum_{i=1}^{m} x_i = 1$ for $x_i \geq 0$, the only solution of $f(v) = v$ is $v = \frac{-2(c_{23} - c_{22}) + 1 - \sqrt{\Delta_v}}{2(c_{22} - 2c_{23} + c_{33})} \in (0,1)$. Here, we denote such a solution as $v^*$. Consequently, the fixed points in (12) can be obtained in the following form:

$$\begin{aligned} x_1^* &= (a_{22} - 2a_{23} + a_{33})(v^*)^2 + 2(a_{23} - a_{22})v^* + a_{22}, \\ x_2^* &= (b_{22} - 2b_{23} + b_{33})(v^*)^2 + 2(b_{23} - b_{22})v^* + b_{22}, \\ x_3^* &= v^*. \end{aligned} \tag{19}$$

(ii) Referring to Remark 2 (iii) and Theorem 2, one may obtain the period-2 periodic points of the operator $W_2$ by solving the function $f^2(v) = v$ with the following roots:

$$v_{\pm}^* = \frac{-2(c_{23} - c_{22}) - 1 \pm \sqrt{\Delta_v - 4}}{2(c_{22} - 2c_{23} + c_{33})},$$

where $v_+^* < v^* < v_-^*$ or $v_-^* < v^* < v_+^*$ and $v_{\pm}^* \neq v^*$.

Note that the fixed points in (17) are also roots for the function $f^2(v) = v$. However, we have eliminated these roots as we aim to find the period-2 periodic points.

Using simple algebra, we may obtain the period-2 periodic points of the system of equations in (15), where we denote $\mathbf{x}_a^* = (x_{1,a}^*, x_{2,a}^*, x_{3,a}^*)$ and $\mathbf{x}_b^* = (x_{1,b}^*, x_{2,b}^*, x_{3,b}^*)$ as

$$\begin{aligned} x_{1,a}^* &= (a_{22} - 2a_{23} + a_{33})(v_-^*)^2 + 2(a_{23} - a_{22})v_-^* + a_{22}, \\ x_{2,a}^* &= (b_{22} - 2b_{23} + b_{33})(v_-^*)^2 + 2(b_{23} - b_{22})v_-^* + b_{22}, \\ x_{3,a}^* &= v_+^*, \end{aligned}$$

and

$$\begin{aligned} x_{1,b}^* &= (a_{22} - 2a_{23} + a_{33})(v_+^*)^2 + 2(a_{23} - a_{22})v_+^* + a_{22}, \\ x_{2,b}^* &= (b_{22} - 2b_{23} + b_{33})(v_+^*)^2 + 2(b_{23} - b_{22})v_+^* + b_{22}, \\ x_{3,b}^* &= v_-^*. \end{aligned}$$

follows:

Thus, the proof is complete.**Proposition 2:** Let $\Delta_v = 4(1 - a_{11})\, a_{33} + (1 - 2a_{13})^2$. For $W_3$, we have:

(i)

$$Fix(W_3) = \{(x_1^*, x_2^*, x_3^*)\}, \text{ where}$$

$$x_1^* = \frac{-2(a_{13} - a_{33}) + 1 - \sqrt{\Delta_v}}{2(a_{33} - 2a_{13} + a_{11})},$$

$$x_2^* = (b_{33} - 2b_{13} + b_{11})(x_1^*)^2 + 2(b_{13} - b_{33})x_1^* + b_{33},$$

$$x_3^* = (c_{33} - 2c_{13} + c_{11})(x_1^*)^2 + 2(c_{13} - c_{33})x_1^* + c_{33}.$$

(ii)

$$Per_2(W_3) = \{(x_{1,a}^*, x_{2,a}^*, x_{3,a}^*), (x_{1,b}^*, x_{2,b}^*, x_{3,b}^*)\}, \text{ where}$$

$$x_{1,a}^* = \frac{-2(a_{13} - a_{33}) - 1 + \sqrt{\Delta_v - 4}}{2(a_{33} - 2a_{13} + a_{11})},$$

$$x_{2,a}^* = (b_{33} - 2b_{13} + b_{11})(x_{1,b}^*)^2 + 2(b_{13} - b_{33})x_{1,b}^* + b_{33},$$

$$x_{3,a}^* = (c_{33} - 2c_{13} + c_{11})(x_{1,b}^*)^2 + 2(c_{13} - c_{33})x_{1,b}^* + c_{33},$$

and

$$x_{1,b}^* = \frac{-2(a_{13} - a_{33}) - 1 - \sqrt{\Delta_v - 4}}{2(a_{33} - 2a_{13} + a_{11})},$$

$$x_{2,b}^* = (b_{33} - 2b_{13} + b_{11})(x_{1,a}^*)^2 + 2(b_{13} - b_{33})x_{1,a}^* + b_{33},$$

$$x_{3,b}^* = (c_{33} - 2c_{13} + c_{11})(x_{1,a}^*)^2 + 2(c_{13} - c_{33})x_{1,a}^* + c_{33}.$$

### *Regularity of Lebesgue QSO with Exponential Measure Generated by 3-Partition with Three Parameters*

**Definition 3:** A QSO $V$ in (2) is a regular if for any initial point $\lambda \in S(X,\mathcal{F})$, the limit

$$\lim_{n\to\infty} V^n\lambda,$$

exists.

**Theorem 3:** If $0 < \Delta < 4$, then, a one-dimensional QSO in (14) is a regular, and if $4 < \Delta < 5$, then, there exists a second-order cycle, where all trajectories incline to this cycle except the stationary trajectory starting with a fixed point.

Definition 3 states that the existence of a QSO limit establishes the regularity of such an operator. The value of the discriminant $\Delta$ must be implied, as stated in Theorem 3, to accurately define the regularity of such a QSO based on the trajectory behaviour. The trajectory behaviour of the Lebesgue QSO with exponential measure generated by 3-partition with three parameters for two distinct cases is discussed in this section, along with their regularity.

Given a function $f$ in (15). Let $(f^2)'(x_1)$ be the function $f^2(x_1)$ first derivative, where

$$(f^2)'(x_1) = 2A(Ax_1^2 + Bx_1 + C)(2Ax_1 + B) + B(2Ax_1 + B),$$

for $A = a - 2b + c$, $B = 2(b - c)$, and $C = c$.

**Proposition 3:** Let $W_2: S^2 \to S^2$ and $0 < \Delta_v < 5$.

(i) If $0 < \Delta_v < 5$, then, an open set $\mathcal{U} \subset S^2$ exists such that $\mathbf{x}^* \in \mathcal{U}$ and for an arbitrary initial point $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in \mathcal{U}$, we have:

$$\lim_{n\to\infty} W_2^{\,n}\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = \left(x_1^*, x_2^*, x_3^*\right).$$

Hence, such an operator $W_2$ is a regular transformation.

(ii) If $4 < \Delta_v < 5$ and $c_{23} < \frac{1}{2} - \sqrt{1-(1-c_{33})c_{22}}$, then, an open set $\mathcal{U} \subset S^2$ exists such that $\mathbf{x}_a^*$, $\mathbf{x}_b^* \in \mathcal{U}$ and for an arbitrary initial point $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in \mathcal{U}$, we have:

$$\lim_{n\to\infty} W_2^{2n}\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \to \begin{cases} \mathbf{x}_a^*, & \text{if } x_3^{(0)} > x_3^* \\ \mathbf{x}^*, & \text{if } x_3^{(0)} = x_3^* \\ \mathbf{x}_b^*, & \text{if } x_3^{(0)} < x_3^* \end{cases},$$

and

$$\lim_{n\to\infty} W_2^{2n+1}\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \to \begin{cases} \mathbf{x}_a^*, & \text{if } x_3^{(0)} < x_3^* \\ \mathbf{x}^*, & \text{if } x_3^{(0)} = x_3^* \\ \mathbf{x}_b^*, & \text{if } x_3^{(0)} > x_3^* \end{cases}.$$

(iii) If $4 < \Delta_v < 5$ and $c_{23} > \frac{1}{2} + \sqrt{1-(1-c_{33})c_{22}}$, then, an open set $\mathcal{U} \subset S^2$ exists such that $\mathbf{x}_a^*$, $\mathbf{x}_b^* \in \mathcal{U}$ and for an arbitrary initial point $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in \mathcal{U}$, we have:

$$\lim_{n\to\infty} W_2^{2n}\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \to \begin{cases} \mathbf{x}_a^*, & \text{if } x_3^{(0)} < x_3^* \\ \mathbf{x}^*, & \text{if } x_3^{(0)} = x_3^* \\ \mathbf{x}_b^*, & \text{if } x_3^{(0)} > x_3^* \end{cases},$$

and

$$\lim_{n\to\infty} W_2^{2n+1}\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \to \begin{cases} \mathbf{x}_a^*, & \text{if } x_3^{(0)} > x_3^* \\ \mathbf{x}^*, & \text{if } x_3^{(0)} = x_3^* \\ \mathbf{x}_b^*, & \text{if } x_3^{(0)} < x_3^* \end{cases},$$

where $\mathbf{x}^* = \left(x_1^*, x_2^*, x_3^*\right)$ is a fixed point and $\mathbf{x}_a^* = \left(x_{1,a}^*, x_{2,a}^*, x_{3,a}^*\right)$ and $\mathbf{x}_b^* = \left(x_{1,b}^*, x_{2,b}^*, x_{3,b}^*\right)$ are periodic point as mentioned previously. Therefore such an operator $W_2$ is a nonregular transformation.

*Proof*

(i) Based on the equation in (16), one may obtain $f'\left(x_3^*\right) = 1 - \sqrt{\Delta}$. This follows that for $0 < \Delta_v < 4$, $x_3^* = \frac{-2(c_{23}-c_{22}) + 1 - \sqrt{\Delta_v}}{2(c_{22} - 2c_{23} + c_{33})} \in (0,1)$ is an attracting hyperbolic fixed point for such a function *f*. Therefore, $x_3^{(n)}$ will converge to $x_3^*$ when $0 < \Delta_v < 4$. The trajectory of the operator $W_2$ on the invariant set $\gamma = \left\{\left(x_1, x_2, x_3\right) \in S^2 : x_3 = x_3^*\right\}$ is as follows:

$$\mathbf{x}^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right),$$

where $x_3^{(n)}$ suffices the equality,

$$x_3^{(n+1)} = \left(c_{22} - 2c_{23} + c_{33}\right)\left(x_3^*\right)^2 + 2\left(c_{23} - c_{22}\right)x_3^* + c_{22}. \quad (20)$$

This indicates that $\lim_{n\to\infty} x_3^{(n)} = x_3^*$. Hence, $\lim_{n\to\infty} W_2^{\,n}\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = \left(x_1^*, x_2^*, x_3^*\right)$ and the operator $W_2$ is regular.

Next, when $4 < \Delta < 5$, it is known from $\left|f'\left(x_3^*\right)\right| = \left|1 - \sqrt{\Delta}\right|$, $x_3^*$ is a repelling hyperbolic fixed point of $f\ (x_3)$. In addition, when $4 < \Delta < 5$, the fixed points $x_{3,a}^*$ and $x_{3,b}^*$ of the function $f^2\ (x_3)$ are attracting. Note that we shall obtain the first derivative of $\left|\left(f^2\right)'\left(x_{3,a}^*\right)\right|$ and $\left|\left(f^2\right)'\left(x_{3,b}^*\right)\right|$ as follows:

$$\left|\left(f^2\right)'\left(x_{3,a}^*\right)\right| = \left|\left(f^2\right)'\left(x_{3,b}^*\right)\right| = 4c_{33}\left(c_{22}-1\right) - 4c_{23}\left(c_{23}-1\right) + 4 = 5 - \Delta.$$

For $4 < \Delta < 5$, it is convenient to obtain $\left|\left(f^2\right)'\left(x_{3,a}^*\right)\right| < 1$ and $\left|\left(f^2\right)'\left(x_{3,b}^*\right)\right| < 1$.

In this case, we shall consider two possible conditions of $c_{23}$. Using the fact that we will get $4 < \Delta < 5$, if and only if $c_{22} \in \left[0, \frac{1}{4}\right]$, $c_{33} > \frac{3}{4(1-c_{22})}$, and $c_{23} < \frac{1}{2} - \sqrt{1-(1-c_{22})c_{33}}$ or $c_{23} > \frac{1}{2} + \sqrt{1-(1-c_{22})c_{33}}$. Now, we show the trajectory behaviour of the operator $W_2$ in (12).

We define the following invariant sets for the operator $W_2$:

$$\gamma = \left\{\left(x_1^*, x_2^*, x_3^*\right) \in S^2 : x_3 = x_3^*\right\},$$

and

$$\Gamma = \left\{\left(x_1, x_2, x_3\right) \in S^2 : x_3 = x_{3,a}^* \text{ or } x_3 = x_{3,b}^*\right\}.$$

Notice that if $\mathbf{x}^{(0)} \in \gamma$, then, $\lim_{n\to\infty} x_3^{(n)} = x_3^*$. If $\mathbf{x}^{(0)} \in \Gamma$, then, we will have $\lim_{n\to\infty} x_3^{(2n)} = x_{3,b}^*$ and $\lim_{n\to\infty} x_3^{(2n+1)} = x_{3,a}^*$ when $x_3^{(0)} = x_{3,b}^*$ for $n = 0,1,2,...$ due to the fact that $x_{3,a}^*$ and $x_{3,b}^*$ are the attracting fixed points of the function $f^2(x_3)$.

Let us consider $\mathbf{x}^{(0)} \in f^2\backslash\gamma$, which directs us to two definite cases, where $x_3^{(0)} < x_3^*$ and $x_3^{(0)} > x_3^*$. One could notice that $x_{3,b}^* < x_3^* < x_{3,a}^*$ when $c_3 < \frac{1}{2} - \sqrt{1-(1-c_{22})c_{33}}$. Therefore, if $x_3^{(0)} < x_3^*$, then, $f\left(x_3^{(0)}\right) > x_3^*$ and $f^2\left(x_3^{(0)}\right) < x_3^*$. As $x_{3,a}^*$ and $x_{3,b}^*$ are attracting fixed points of $f^2(x_3)$, consequently, we may have $\lim_{n\to\infty} f^{2n}\left(x_3^{(0)}\right) = x_{3,b}^*$ and $\lim_{n\to\infty} f^{2n+1}\left(x_3^{(0)}\right) = x_{3,a}^*$ for $n = 0,1,2,...$

Correspondingly, if $x_3^{(0)} > x_3^*$, one will get $f\left(x_3^{(0)}\right) < x_3^*$ and $f^2\left(x_3^{(0)}\right) > x_3^*$, which gives us $\lim_{n\to\infty} f^{2n}\left(x_3^{(0)}\right) = x_{3,a}^*$ and $\lim_{n\to\infty} f^{2n+1}\left(x_3^{(0)}\right) = x_{3,b}^*$ for $n = 0,1,2,...$

(ii) The establishment of the previous proof allows us to describe the operator $W_2$ trajectory behaviour when $4 < \Delta < 5$ and $c_{23} > \frac{1}{2} + \sqrt{1-(1-c_{22})c_{33}}$. Be mindful that if $\mathbf{x}^{(0)} \in \gamma$, then, $\lim_{n\to\infty} x_3^{(n)} = x_3^*$. Subsequently, when $c_{23} > \frac{1}{2} + \sqrt{1-(1-c_{22})c_{33}}$, we have $x_{3,a}^* < x_3^* < x_{3,b}^*$. For the case $x_3^{(0)} < x_3^*$, we may conclude that $\lim_{n\to\infty} f^{2n}\left(x_3^{(0)}\right) = x_{3,a}^*$ and $\lim_{n\to\infty} f^{2n+1}\left(x_3^{(0)}\right) = x_{3,b}^*$ for $n = 0,1,2,...$ since $f\left(x_3^{(0)}\right) > x_3^*$ and $f^2\left(x_3^{(0)}\right) < x_3^*$.

Accordingly, when $x_3^{(0)} < x_3^*$, one may obtain $f\left(x_3^{(0)}\right) < x_3^*$ and $f^2\left(x_3^{(0)}\right) > x_3^*$. Thus, $\lim_{n\to\infty} f^{2n}\left(x_3^{(0)}\right) = x_{3,b}^*$ and $\lim_{n\to\infty} f^{2n+1}\left(x_3^{(0)}\right) = x_{3,a}^*$ for $n = 0,1,2,...$ It is shown that the operator $W_2$ is nonregular when $4 < \Delta < 5$. Hence, we conclude the proof.

Given that the operator $W_2$ can be a regular or a nonregular transformation, it is not difficult to describe the trajectory behaviour of the operator $W_3$ as follows:

**Corollary 1:** The operator $W_3$ in (13) can be a regular or a nonregular transformation.

## Conclusions

The Lebesgue QSO with exponential measure generated with 3-partition with three parameters for two different cases of the parameters, i.e., $\mu_{11} = \mu_{22} = \mu_{22} \neq \mu_{13} = \mu_{23} \neq \mu_{33}$ and $\mu_{22} = \mu_{33} = \mu_{23} \neq \mu_{12} = \mu_{13} \neq \mu_{11}$ have been constructed. It is known that such QSOs defined on a continuous state space can be reduced to a finite-dimensional setting, which corresponds to the partitions. The two-dimensional QSO in (12) and (13) can be further reduced to a one-dimensional simplex due to its reducible form of the system of equations.

This article presented the dynamics of such QSOs by providing a comprehensive discussion of the existence of a unique fixed point, where the fixed point may be an attracting or a non-attracting fixed point according to the value of discriminant denoted by $\Delta_v$. It is proven that for such QSOs, one may obtain $0 < \Delta_v < 5$ for the quadratic function derived from the reducible two-dimensional QSOs. It follows that when $0 < \Delta_v < 4$, the one-dimensional QSO has an attractive fixed point, which demonstrates a strong limit of the QSO. Moreover, when $4 < \Delta_v < 5$, the fixed point is non-attracting, it leads to the presence of the second-order cycle, where the strong limit of the QSO does not exist. We showed that the strong limit indicates that such a QSO is a regular transformation while a second-order cycle implies a nonregular transformation.

The study of QSO generated by finite, measurable partitions defined on infinite state space on population genetics and dynamical systems would significantly contribute to sustainable management practice.

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**Conflict of Interest Statement**

The authors agree that this research was conducted in the absence of any self-benefits or commercial or financial conflicts and declare the absence of conflicting interests with the funders.

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